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A Nonlinear Finite Element Formalism for Modelling Flexible and Soft Manipulators

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***Abstract*— This paper presents a nonlinear finite element formalism for modelling the dynamics of flexible manipulators using the special Euclidean group *SE*(3) framework. The method is based on a local description of the motion variables, and results in a singularity−free formulation which exhibits important advantages regarding numerical implementation. The motivation behind this work is the development of a new class of model−based control systems which may predict and thus avoid the deformations of a real flexible mechanism. Finite element methods based on the geometrically exact beam theory have been proven to be the most accurate to account for flexibility: in this paper we highlight the key aspects of this formulation deriving the equations of motion of a flexible constrained manipulator and we illustrate its potential in robotics through a simple case study, the dynamic analysis of a two−link manipulator, simulating different model assumptions in order to emphasize its real physical behavior as flexible mechanism.**

***Index Terms*— Flexible manipulators, soft robots, differential geometry, motion formalism, nonlinear finite element, robot simulation.**

I. Introduction

Flexible manipulators refer to robot manipulators having components with mechanical flexibility, either concentrated at the joints or distributed along the links [1]. With respect to classical rigid systems, flexible link manipulators have the potential advantages of (i) greater payload−to−mass ratio; (ii) higher operational speed; (iii) exploiting new composite materials in designing; (iv) lower energy consumption; (v) better manoeuvrability and transportability; (vi) safer operations due to reduced inertia [2]. These potential advantages result in a greater complicated mathematical model of flexible arms suitable to catch all dynamic aspects of distributed flexibility, as well as a greater complicated motion control of the overal manipulator, since control laws, in addition to the tasks of regulation or tracking, have also to avoid incipient oscillations due to previous motion.

Currently, the most diffused approach to model a flexible arm is to use the classical Euler−Bernoulli beam model and discretize the resulting partial differential equations (PDE) with the assumed mode method (AMM) [3]. This strategy is the natural extension of the rigid multibody dynamics to the context of flexible multibody systems: small elastic deformations are superimposed to the classical overall rigid body motion of a floating frame attached to each flexible body [4]. Despite its proven efficiency for control purposes [5], this approach is limited to flexible arms rotating at modest angular rate (so that centrifugal stiffening can be ignored) and connected only with revolute joints, with the assumption of light damping and linear elasticity, thus small deflections [1].

In this paper we propose a nonlinear finite element formalism for modelling flexible manipulators. Considering the motion of a component as a whole, the formulations based on the geometrically exact nonlinear beam theory are the most general and accurate to account for flexibility [6]. Finite element procedures provide several features of interest in modelling multibody systems: (1) a manipulator can be modelled with both rigid and flexible elements; (2) there are no differences in modelling serial or parallel mechanism; (3) all kinds of low pair and high pair joints (which can be also passive) may be modelled for connecting bodies. The main drawback of finite element (FE) procedures is the high computational cost of the associated numerical methods to solve the equations of motion of the overall system. The current work aims at developing a mathematical formulation for flexible manipulators which enjoys the advantages of FE methods and provides at the same time a cost effective computation. Using the special Euclidean group *SE*(3) formalism, it is based on a geometrically exact approach which results in reduced non−linearities and in efficient and robust numerical methods for integrating the equations of motion. The main purpose of the present paper is to show a general theoretical framework useful for modelling and simulating flexible and rigid manipulators. This formulation has led to the development of a very simple and modular simulator for the dynamics of both rigid and flexible multibody systems, which in the future will be extended with motion planning and control capabilities.

The rest of this paper is organized as follows. In the second part of this section we provide a brief *state−of−the−art* in modelling of flexible manipulators. In Section II we present the general approach of the finite element formulation, the *motion formalism*. In Section III we derive the equations of motion of a constrained flexible manipulator. Section IV is related to the benchmark test. Section V concludes the paper and discusses future developments.

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A. Related work

Since Book presented his pioneering work [3], a lot of work has been done in the context of flexible manipulators. Wasfy and Noor [7] proposed a comprehensive review of the existing modelling formulations in the more general context of flexible multibody systems. Their classification is based on the frame wherein the flexibility and the large amplitude motions are referred: floating frame of reference (FFR), corotational frame (CF), inertial frame (IF). De Luca and Book [1] proposed a classification of modelling techniques for flexible link arms based on the approach for discretizing the elements: lumped−element, finite−element, assumed mode and transfer matrix models. Theodore and Ghosal in [8] compared the assumed mode models with the finite elements models.
Starting from the pioneer works of Simo and Vu−Quoc [9], [10], several nonlinear beam models are available in the literature, see *e.g.* [11], [12]. Here, we refer to the beam model developed by Sonneville *et. al* in [13], as it offers appealing properties towards the reduction of computational cost needed to accomodate non−linear finite element in control applications.

II. The Motion Formalism

The presented formulation, used for in the context of multibody systems in [14], is based upon *nodal absolute variables* for the description of large amplitude motions of the bodies and *kinematic joint relative variables* for the connection between bodies. These variables are represented by material frames, which belong to the Special Euclidean Lie Group $SE(3)$ and can be expressed as 4 *x* 4 homogeneous transformation matrices $\mathbf{H}$ as follows

$$\mathbf{H}=\begin{bmatrix}\mathbf{R} & \mathbf{x}\\ \mathbf{0}_{1\times3} & 1\end{bmatrix} \tag{1}$$

where $\mathbf{R}$ is a 3×3 rotation matrix and $\mathbf{x}$ is a 3×1 vector. For the absolute variables, the rotation matrix and the position vector describe the orientation and position of the node with respect to the inertial frame, while for the relative variables they represent the relative rotation and the relative displacement inside the joint.
The kinematics of a general multibody system is provided by successive frame transformations: a frame is attached to each rigid body or to the cross−section of each beam element (a rigid body is represented by a unique node at its mass center whereas a flexible body is represented by a finite set of nodes). Moreover, further nodes are introduced to allow the description of boundary conditions or to specify a kinematic joint between two bodies. Figure 1 shows, within this formalism, the geometric description of a general manipulator including both rigid and flexible links.

III. Equations of Motion

In this section we obtain the equations of motion (EoM) of a free rigid body, a beam element (in its continuum and discretized form) and a constrained flexible manipulator. A section is dedicated to the kinematics of joints.

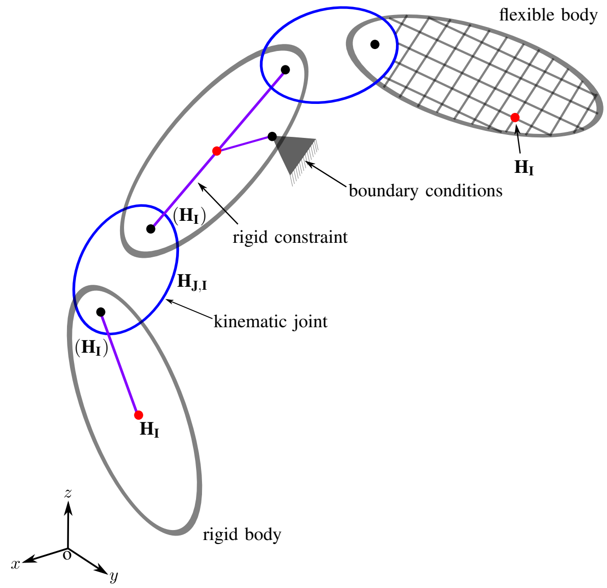

Fig. 1. General description of a robotic manipulator: $\mathbf{H}_I$ are nodal frames, $\mathbf{H}_{J,I}$ are relative transformations and $(\mathbf{H}_I)$ are optional frames, used to specify boundary conditions or kinematic joints.

A. Rigid body

The kinematics of rigid bodies is described using frames, elements of the special Euclidean group $SE(3)$: each rigid body is represented by a material frame located at its center of gravity $(\mathbf{H}_I)$. In this context we use left invariant vector fields to express the derivatives, which involve material−frame elements, that are introduced as follows

$$d(\mathbf{H}_I)=\mathbf{H}_I\tilde{\mathbf{h}}_I=\begin{bmatrix}\mathbf{R}_I & \mathbf{x}_I\\ \mathbf{0}_{1\times3} & 1\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{h}}_{I,\Omega} & \mathbf{h}_{I,U}\\ \mathbf{0}_{1\times3} & 1\end{bmatrix} \tag{2}$$

where $\mathbf{h}_I=[\mathbf{h}_{I,U}^T\,\mathbf{h}_{I,\Omega}^T]^T$ is a six−dimensional vector. $\tilde{(\cdot)}$ is a linear operator which maps a six−dimensional vector into a 4×4 matrix as in Eq. 2 (for the Lie algebra, see [15]). The two part 3×1 vectors $(\mathbf{h}_{I,U})$ and $(\mathbf{h}_{I,\Omega})$ are related to the position and the rotation part, respectively. Here, $\tilde{\mathbf{h}}_{I,\Omega}$ is the skew−symmetric matrix formed by the three components of $\mathbf{h}_{I,\Omega}$ as

$$\tilde{\mathbf{h}}_{I,\Omega}=\begin{bmatrix}0 & -\mathbf{h}_{I,\Omega_3} & \mathbf{h}_{I,\Omega_2}\\ \mathbf{h}_{I,\Omega_3} & 0 & -\mathbf{h}_{I,\Omega_1}\\ -\mathbf{h}_{I,\Omega_2} & \mathbf{h}_{I,\Omega_1} & 0\end{bmatrix} \tag{3}$$

As an alternative, it is possible to consider the derivatives as $d(\mathbf{H}_I)=\tilde{\mathbf{h}}_I\mathbf{H}_I$, with $\mathbf{h}_I$ representing the derivatives in the inertial frame [16]. However, the left invariant approach leads to intrinsic EoM which, as we will see in Sec. III-E, offer computational advantages. A particular case of derivative is an arbitrary variation, which is expressed as

$$\delta\mathbf{H}_I=\mathbf{H}\widetilde{\delta\mathbf{h}}_I \tag{4}$$

If we consider derivatives in $SE(3)$ with respect to time (the time derivative of $(\cdot)$ will be indicated in the following as $\dot{(\cdot)}$), the kinematics of a rigid body can be expressed as

$$\dot{\mathbf{H}}=\mathbf{H}\tilde{\mathbf{v}} \tag{5}$$

with $\mathbf{v} = [\mathbf{v}_U^T \, \mathbf{v}_\Omega^T]^T$ is the six−dimensional vector of material frame velocities.
Hamilton's principle, in the case of a free rigid body without external forces, takes the form of

$$\delta\left(\int_{t_i}^{t_f} K(\mathbf{v})\, dt\right) = 0 \tag{6}$$

where $K(\mathbf{v}) = \frac{1}{2}\mathbf{v}^T\mathbf{M}\mathbf{v}$ is the kinetic energy and $\mathbf{M}$ is the 6×6 mass matrix of the rigid body, which is expressed in the material frame and therefore constant. The equilibrium equations are given by

$$\mathbf{M}\dot{\mathbf{v}} - \hat{\mathbf{v}}^T\mathbf{M}\mathbf{v} = \mathbf{0}_{6\times1} \tag{7}$$

where the $\hat{(\cdot)}$ operator maps a 6×1 vector into a 6×6 matrix as

$$\hat{\mathbf{v}}_I = \begin{bmatrix} \tilde{\mathbf{v}}_{I,\Omega} & \tilde{\mathbf{v}}_{I,U} \\ \mathbf{0}_{3\times3} & \tilde{\mathbf{v}}_{I,\Omega} \end{bmatrix} \tag{8}$$

Note that EoM, *i.e.* (7) together with (5), are expressed in the local frame attached to the rigid body. Eq. 7 are called intrinsic since they do not depend on the position and orientation of the rigid body with respect to the inertial frame, namely they are invariant with respect to a superimposed Euclidean transformation.

B. Geometrically exact beam

A beam can be seen as a cross−section field along a reference curve. In this context we refer to the geometrically exact nonlinear beam element developed in [13]. This beam element accounts for shear and torsion deformations remaining within the context of small strains.

Continuous formulation: The kinematics of the beam is described by

$$\mathbf{H}(\alpha,t) = \begin{bmatrix} \mathbf{R}(\alpha,t) & \mathbf{x}(\alpha,t) \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix} \tag{9}$$

where α refers to the spatial coordinate along the beam reference curve. Introducing left invariant vectors fields, the velocity and the deformation field along the beam can be defined as

$$\frac{\partial\mathbf{H}(\alpha,t)}{\partial t} = \dot{\mathbf{H}}(\alpha,t) = \mathbf{H}(\alpha,t)\tilde{\mathbf{v}}(\alpha,t) \tag{10}$$

$$\frac{\partial\mathbf{H}(\alpha,t)}{\partial \alpha} = \mathbf{H}'(\alpha,t) = \mathbf{H}(\alpha,t)\tilde{\mathbf{f}}(\alpha,t) \tag{11}$$

where $\tilde{\mathbf{f}}(\alpha,t)$ is a material frame deformation gradient. It can be split into two parts as follows

$$\mathbf{f}(\alpha,t) = \mathbf{f}^0(\alpha) + \boldsymbol{\varepsilon}(\alpha,t) \tag{12}$$

In this formula, $\mathbf{f}^0$ refers to the values of gradient in the reference configuration, while $\boldsymbol{\varepsilon}$ are the six deformation measures of the beam. In order to derive the EoM for a beam element, Hamilton's principle states that

$$\delta\left(\int_{t_i}^{t_f} (K(\mathbf{v}) - V_{int}(\boldsymbol{\varepsilon}))\, dt\right) = 0 \tag{13}$$

where $K(\boldsymbol{\nu}) = \frac{1}{2}\int_L \mathbf{v}^T\mathbf{M}\mathbf{v}\, d\alpha$ and $V_{int}(\boldsymbol{\varepsilon}) = \frac{1}{2}\int_L \boldsymbol{\varepsilon}^T\mathbf{K}\boldsymbol{\varepsilon}\, d\alpha$ are respectively the kinetic energy and the strain internal energy along the beam of length L. $\mathbf{M}$ and $\mathbf{K}$ are respectively the mass matrix and stiffness matrix of the cross-sections. $\mathbf{K}$ is a 6×6 matrix relating the 6 cross−section resultants (forces and moments) and the deformations. In the abscence of external forces, the equilibrium equations for a beam are given by

$$\mathbf{M}\dot{\mathbf{v}} - \hat{\mathbf{v}}^T\mathbf{M}\mathbf{v} + \mathbf{K}\boldsymbol{\varepsilon}' - \hat{\mathbf{f}}^T\mathbf{K}\varepsilon = \mathbf{0}_{6\times1} \tag{14}$$

$$\mathbf{v}' - \dot{\boldsymbol{\varepsilon}} - \hat{\mathbf{v}}\mathbf{f} = \mathbf{0}_{6\times1} \tag{15}$$

Eqs. 10, 11, 14 and 15 constitute the EoM for the geometrically nonlinear beam element in the strong form. These equations are intrinsic, resulting in a second−order nonlinearity only.

Discretized formulation: The continuous beam is here discretized with two nodes, A and B, placed at the beginning and at the end of the element, each one providing six degrees of freedom. The kinematics of the beam involves the two nodal transformation matrices

$$\mathbf{H}_A = \begin{bmatrix} \mathbf{R}_A & \mathbf{x}_A \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix} \quad \mathbf{H}_B = \begin{bmatrix} \mathbf{R}_B & \mathbf{x}_B \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix} \tag{16}$$

The spatial discretization along the reference curve of the beam is introduced by means of the following $SE(3)$ consistent interpolation field

$$\mathbf{H}(\alpha) = \mathbf{H}_A \exp_{SE(3)}\left(\frac{\alpha}{L}\tilde{\mathbf{d}}\right) \tag{17}$$

where $\alpha \in [0,L]$ assumes the extreme values in the two nodes, and $\mathbf{d}$, referred to the relative configuration vector, is a six-dimensional vector defined as $\tilde{\mathbf{d}} = \log_{SE(3)}(\mathbf{H}_A^{-1}\mathbf{H}_B)$. Geometrically, the interpolation field is represented by an helix. In the absence of external forces, the spatial discretization leads to the following EoM

$$\dot{\mathbf{H}}_A = \mathbf{H}_A\tilde{\mathbf{v}}_A \tag{18}$$

$$\dot{\mathbf{H}}_B = \mathbf{H}_B\tilde{\mathbf{v}}_B \tag{19}$$

$$\mathbf{M}^L(\mathbf{d})\dot{\mathbf{v}}_{AB} + \mathbf{C}^L(\mathbf{d},\mathbf{v}_{AB})\mathbf{v}_{AB} + \frac{1}{L}\mathbf{P}(\mathbf{d})^T\mathbf{K}^L\boldsymbol{\varepsilon} = \mathbf{0}_{12\times1} \tag{20}$$

where $\boldsymbol{\varepsilon} = (\mathbf{d} - \mathbf{d}^0)/L$ is the discretized strain vector, $\mathbf{v}_{AB} = [\mathbf{v}_A^T \, \mathbf{v}_B^T]^T$ is the vector of velocities, $\mathbf{M}^L$ and $\mathbf{K}^L$ are the mass and the stiffness matrix of the element, $\mathbf{C}^L(\mathbf{d},\mathbf{v}_{AB})\mathbf{v}_{AB}$ is a term accounting for the gyroscopic forces. The discretization process preserves the intrinsic nature of the equations and leads to low order non-linearities.

C. Kinematic joint

In the $SE(3)$ formalism, the relashionship between two nodes A and B which are connected by a kinematic joint I can be expressed as

$$\mathbf{H}_B = \mathbf{H}_A\mathbf{H}_{J,I} \tag{21}$$

where $\mathbf{H}_{J,I}$ is a transformation matrix that describes the relative motion between the two nodes due to joint I. The restricted relative motions leave $k_I < 6$ degrees of freedom to the joint, and accordingly, the transformation matrices $\mathbf{H}_{J,I}$ belong to a subset of $SE(3)$ [17]. As for the nodal frames, the derivative of $\mathbf{H}_{J,I}$ can be expressed as

$$d(\mathbf{H}_{J,I}) = \mathbf{H}_{J,I}\widetilde{\mathbf{A}_I\mathbf{h}}_{j,I} \tag{22}$$

TABLE I

MATRIX **A** FOR THE LOW−PAIR JOINTS.

Joint	Rigid Constraint	Revolute	Prismatic	Screw (pitch p)	Cylindrical	Planar	Spherical
A	$\begin{bmatrix}\mathbf{0}_{3x1}\\ \mathbf{0}_{3x1}\end{bmatrix}$	$\begin{bmatrix}\mathbf{0}_{3x1}\\ \mathbf{n}\end{bmatrix}$	$\begin{bmatrix}\mathbf{n}\\ \mathbf{0}_{3x1}\end{bmatrix}$	$\begin{bmatrix}p\mathbf{n}\\ \mathbf{n}\end{bmatrix}$	$\begin{bmatrix}\mathbf{0}_{3x1} & \mathbf{n}\\ \mathbf{n} & \mathbf{0}_{3x1}\end{bmatrix}$	$\begin{bmatrix}\mathbf{n}_1 & \mathbf{n}_2\\ \mathbf{0}_{3x1} & \mathbf{0}_{3x1}\end{bmatrix}$	$\begin{bmatrix}\mathbf{0}_{3x1} & \mathbf{0}_{3x1} & \mathbf{0}_{3x1}\\ \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3\end{bmatrix}$
Dimension	−	1	1	1	2	2	3
Lie Subgroup	−	$SO(2)$	$\Re$	H_p	$SO(2)\ x\Re$	$\Re^2$	$SO(3)$

where $\mathbf{h}_{j,I}$ is a $k_I \times 1$ vector representing the relative degrees of freedom. $\mathbf{A}_I$ is a $6 \times k_I$ matrix representing the relative motions inside the joint. For low pair joints (see TABLE I), $\mathbf{A}_I$ is constant. Note that the high pair joints can be obtained as combinations of low pair joints.

D. Constrained flexible manipulators

According to Hamilton's principle and following a Lagrange multiplier method, the actual trajectory of a system between two time instants t_i and t_f is such that the variation of the augmented action integral is time−invariant provided that the initial and final configurations are fixed, *i.e.*:

$$\delta\left(\int_{t_i}^{t_f} (K(\mathbf{H},\mathbf{v}) - V_{int}(\mathbf{H}) - V_{ext}(\mathbf{H}) - \lambda^T \phi(\mathbf{H}))\, dt\right) = 0 \qquad (23)$$

where:

- $K(\mathbf{H},\mathbf{v})$ is the kinetic energy of the overall system;
- $V_{int}(\mathbf{H})$ is the potential energy due to internal forces;
- $V_{ext}(\mathbf{H})$ is the potential energy due to external forces;
- λ are the Lagrange multipliers associated with the kinematic constraints $\phi(\mathbf{H}) = 0$.

Considering a general manipulator with M nodal frames and m kinematics joints, with $k = k_1 + \cdots + k_m$ degrees of freedom of all joints, the general EoM of a flexible constrained manipulator are provided by the following three set of second−order differential−algebraic equations (DAE). Eq. 24 are the kinematic compatibility equations, Eq. 25 are the equilibrium equations, Eq. 26 are the kinematic constraint equations to take into account the presence of joints.

$$\dot{\mathbf{H}} = \mathbf{H}\tilde{\mathbf{v}} \qquad (24)$$

$$\mathbf{g}_{ine}(\mathbf{H},\mathbf{v},\dot{\mathbf{v}}) + \mathbf{g}_{int}(\mathbf{H}) + (\phi_q(\mathbf{H})\mathbf{A})^T\lambda - \mathbf{g}_{ext}(\mathbf{H}) = \mathbf{0}_{(6M+k)\times 1} \qquad (25)$$

$$\phi(\mathbf{H}) = \mathbf{0}_{6m\times 1} \qquad (26)$$

In the following, each of these equations will be analyzed.

a) Kinematics: The kinematics of the flexible manipulator is expressed by a unified matrix notation to treat at the same time the nodal variables and the kinematic joints. The actual configuration of the system is properly described using the following $(6M+6m) \times (6M+6m)$ block diagonal matrix

$$\mathbf{H} = \mathrm{diag}(\mathbf{H}_1, \ldots, \mathbf{H}_M, \mathbf{H}_{J,1}, \ldots, \mathbf{H}_{J,m}) \qquad (27)$$

wherein $\mathbf{H}_I$, $I = 1,\ldots,M$ are the body−attached frames, referred as nodal frames, and $\mathbf{H}_{J,I}$, $I = 1,\ldots,m$ are the matrices representing the relative transformations due to joints. Combining the velocities of each frame into $\mathbf{v}$, we have

$$\tilde{\mathbf{v}} = \mathrm{diag}(\tilde{\mathbf{v}}_1, \ldots, \tilde{\mathbf{v}}_M, \widetilde{\mathbf{A}_1\mathbf{v}}_{j,1}, \ldots, \widetilde{\mathbf{A}_m\mathbf{v}}_{j,m}) \qquad (28)$$

b) Dynamics: The equilibrium equations are obtained applying the variation calculus on the Hamilton's principle in Eq. 23. In the resulting dynamic equilibrium equations 25, $\mathbf{g}_{ine}$, $\mathbf{g}_{int}$ and $\mathbf{g}_{ext}$ are respectively the inertia, the internal and the external forces, while ϕ_q is the contraint gradient explained in the next paragraph. Note that the inertia forces here include also the gyroscopic forces.

c) Constraint equations: In order to account for the presence of joints in the Hamilton's principle, a constraint equation vector $\phi(\mathbf{H}) = [\phi_1 \ldots \phi_m]^T$ is introduced where, for each joint I, six constraints are given by

$$\phi(\mathbf{H}_A, \mathbf{H}_B, \mathbf{H}_{J,I}) = \mathrm{vect}_{SE(3)}(\mathbf{H}_A, \mathbf{H}_B, \mathbf{H}_{J,I}) = \mathbf{0}_{6\times 1} \qquad (29)$$

with the vectorial map defined as

$$\mathrm{vect}_{SE(3)}(\mathbf{H}) = \begin{bmatrix}\mathbf{x}\\ \boldsymbol{\psi}\end{bmatrix} \qquad (30)$$

with $\tilde{\boldsymbol{\psi}} = (\mathbf{R} - \mathbf{R}^T)/2$. The vectorial map introduces systematically six constraints for a kinematic joint or a rigid constraint: it imposes that the relative displacements and rotations contained in $\mathbf{H}_{J,I}$ are exactly the relative configuration between nodes A and B. The constraint contribution to the EoM in Eq. 25 involves the constraint gradient ϕ_q, defined from the directional derivative of the constraints, *i.e.*,

$$\delta(\phi) = D\phi \cdot \widetilde{\mathbf{A}\delta\mathbf{h}} = \phi_q\mathbf{A}\delta\mathbf{h} \qquad (31)$$

where, defining $\mathbf{A} = \mathrm{diag}(\mathbf{I}_{6\times 6}, \mathbf{I}_{6\times 6}, \mathbf{A}_I)$ and $\delta\mathbf{h} = \begin{bmatrix}\delta\mathbf{h}_A^T & \delta\mathbf{h}_B^T & \delta\mathbf{h}_{j,I}^T\end{bmatrix}^T$,

$$\phi_q\mathbf{A} = \begin{bmatrix}\mathrm{Ad}_{\mathbf{H}_{J,I}^{-1}} & -\mathbf{I}_{6\times 6} & \mathbf{A}_I\end{bmatrix} \qquad (32)$$

in which Ad is the adjoint representation given by

$$\mathrm{Ad}_{\mathbf{H}_{J,I}^{-1}} = \begin{bmatrix}\mathbf{R}_{J,I}^T & -\mathbf{R}_{J,I}^T\tilde{\mathbf{x}}_{J,I}\\ \mathbf{0}_{3\times 3} & \mathbf{R}_{J,I}^T\end{bmatrix} \qquad (33)$$

The constraint gradient at equilibrium only depends on the relative configuration, and not on the overall motion of frames A and B. This means that the non−linearity of the formulation is only caused by local motions.

E. Time integration method

In order to solve the equations of motion in Eqs. 24, 25 and 26 a time integration method is needed. Brüls in [18] proposed a version of the generalized−α scheme: this method has a proven second−order convergence rate. The integration method relies on the following equations, which are the discretized form of the EoM 24, 25 and 26:

$$\mathbf{H}_{n+1} = \mathbf{H}_n \exp(\mathbf{n}) \tag{34}$$

$$\mathbf{g}_{ine}(\mathbf{H}_{n+1}, \mathbf{v}_{n+1}, \dot{\mathbf{v}}_{n+1}) + \mathbf{g}_{int}(\mathbf{H}_{n+1}) + \mathbf{A}^T \phi_q^T(\mathbf{H}_{n+1})\lambda - \mathbf{g}_{ext}(\mathbf{H}_{n+1}) = \mathbf{0}_{(6M+k)\times 1} \tag{35}$$

$$\phi(\mathbf{H}_{n+1}) = \mathbf{0}_{6m\times 1} \tag{36}$$

and on the time integration formulas:

$$\mathbf{n} = h\mathbf{v}_n + (0.5-\beta)h^2\mathbf{a}_n + \beta h^2 \mathbf{a}_{n+1} \tag{37}$$

$$\mathbf{v}_{n+1} = \mathbf{v}_n + (1-\gamma)h\mathbf{a}_n + \gamma h \mathbf{a}_{n+1} \tag{38}$$

$$\mathbf{a}_{n+1} = \frac{1}{(1-\alpha_m)}((1-\alpha_f)\dot{\mathbf{v}}_{n+1} + \alpha_f \dot{\mathbf{v}}_n - \alpha_m \mathbf{a}_n) \tag{39}$$

where n is the time step and h the time step size and the numerical parameter are defined in terms of a singular parameter, $\rho \in [0,1]$, which controls the numerical damping

$$\alpha_m = \frac{2\rho-1}{\rho+1}; \quad \alpha_f = \frac{\rho}{\rho+1}; \quad \gamma = \frac{3-\rho}{2(\rho+1)}; \quad \beta = \frac{1}{(\rho+1)^2}. \tag{40}$$

The exponential map appearing in Eq. 34 for a multibody system is defined as follows

$$\exp(\mathbf{n}) = \mathrm{diag}(\exp_{SE(3)}(\mathbf{n}_1), \ldots, \exp_{SE(3)}(\mathbf{n}_M), \exp_{SE(3)}(\mathbf{n}_{J,1}), \ldots, \exp_{SE(3)}(\mathbf{n}_{J,m})) \tag{41}$$

where $\mathbf{n}_I$ is a six−dimensional vector of motion increment. Thus, the time integration method is such that only the incremental motion between n and $n+1$ is parametrized, which does not introduce any non-lineaity in the discretized equilibrium and constraint equations, as opposed to methods relying on a global parametrization of the rotations variables. The discretized EoM (34)−(36) are nonlinear, and are solved at each time step by a Newton iterative procedure. Denoting a finite variation due to Newton procedure as $\Delta(\cdot)$, the linearization of the equations leads to the following linear problem

$$\begin{bmatrix}\Delta\mathbf{r}\\ \Delta\phi\end{bmatrix} = \mathbf{S}_T \begin{bmatrix}\Delta\mathbf{n}_{n+1}\\ \Delta\lambda_{n+1}\end{bmatrix} \tag{42}$$

where Eq. 35 is denoted as $\mathbf{r} = \mathbf{0}$ and

$$\mathbf{S}_T = \begin{bmatrix} \frac{1-\alpha_m}{\beta h^2(1-\alpha_f)}\mathbf{M} + \frac{\gamma}{\beta h}\mathbf{C}_T + \mathbf{K}_T\mathbf{T} & \phi_q^T \\ \phi_q^*\mathbf{T} & \mathbf{0} \end{bmatrix} \tag{43}$$

is the iteration matrix, $\mathbf{M}, \mathbf{C}_T, \mathbf{K}_T$ are the mass, damping and stiffness tangent matrices obtained by linearizing Eq. 35 with respect to $\mathbf{v}$, $\dot{\mathbf{v}}$ and $\mathbf{n}$, $\mathbf{T}$ is the tangent operator of the exponential map. As a consequence of the present framework, the nonlinearities are much lower than standard finite element formulations and the cost of Newton iterative procedure, which is usually the most expensive part of a finite element solver, will be accordingly much lower.

IV. Benchmark test

A two−link manipulator is considered as a case study. To show how it is important to take into account the real behavior of the manipulators even in a simple case, we propose four differents models, referred in the following as: RR (both rigid links), RF (first rigid, second flexible), FR (first flexible, second rigid) and FF (both flexible links).

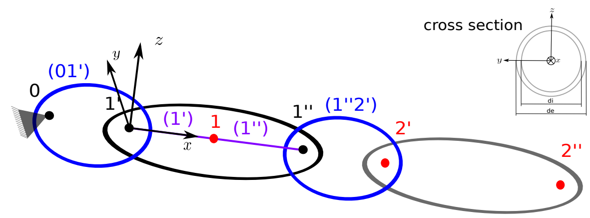

Fig. 2. The two−link manipulator in the rigid-flexible configuration

Figure 2 describes the geometry of the RF model within this formalism. The other models are obtained similarly.

A. Description of the model

In the following we describe only the RF model. The rigid body is described by node 1 defined in its mass center, the flexible body by two nodes 2' and 2" defined in its extremity. The nodes 1' and 1" are introduced to describe respectively the kinematic joints (01') and (1"2') and are both connected with the node 1 by means of rigid constraints (1') and (1"). The node 0 is a fixed node, *i.e.* clamped, meaning that this manipulator is fixed.

B. Simulations

In the simulation case, the first revolute joint (01') describes a circonference in the plane xy in 2π s. During this time, the motion of the overall mechanism is observed. The length of each arm is $l = 1.5$ m, the cross section is a circular ring with $d_i = 2.7\times 10^{-2}$ m and $d_e = 3\times 10^{-2}$ m, the material is aluminium alloy 6061. These parameters have been chosen to highlight the flexible behavior of the manipulator in this configuration. The arms are along the x−direction in the initial straight configuration. The system is subjected to gravity in the z−direction. The simulations have been performed with $h = 1\times 10^{-3}$, 1×10^{-2} and 1×10^{-1} s. The simulations plot, in the inertial reference frame and for each model (RR, RF, FR, FF) the three components of displacements of the last node of the manipulator, in the case $h = 1\times 10^{-2}$ s (Figure 3). This benchmark test has been chosen since only the models with flexibility shows a non−zero displacement along the z−direction. In particular, Figure 3(c) shows displacements in the order of 10^{-1} m for the FF model: this behavior may be attenuated using different cross−sections. As we expected, the flexible systems present a slight delay in response, and the displacements with respect to the other models are similar only at the beginning.

Table II reports the computational time required for solving the EoM associated with the different models on a Intel ® Core™ i7-4910MQ CPU (quad-core 2.50 GHz, Turbo 3.50 GHz), 32 Gb RAM 1600MHz DDR3L, NVIDIA®Quadro®K2100M w/2GB GDDR5 VGA machine, running Ubuntu 14 64 bits. Even if for the rigid case the FE method, due to constraints, will be always slower than minimal coordinate formulations, the computational time for the flexible case is promising, expecially for more refined models (FF2,5,10 refer to model with 2,5,10 beam elements for each arm).

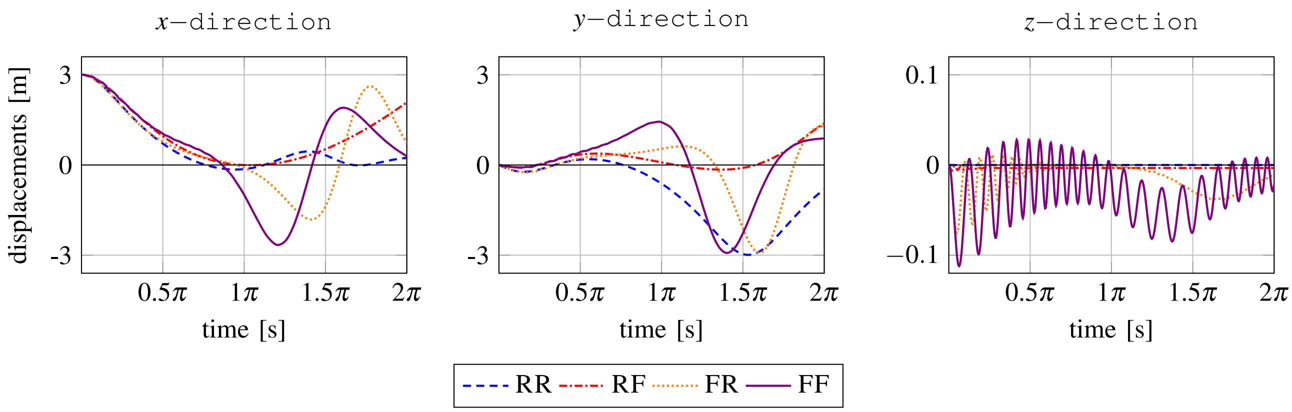

Fig. 3. Displacements of the tip−position (node 2”) of the manipulator in the inertial frame for the four different models, in the case $h = 1 \times 10^{-2}$ s.

TABLE II

COMPUTATIONAL TIME FOR SOLVING THE DIFFERENT MODELS.

h [s]	RR	RF	FR	FF	FF2	FF5	FF10
10^{-3}	149.12	175.96	167.47	156.09	169.05	176.11	198.41
10^{-2}	14.26	15.64	15.71	15.46	17.18	19.04	21.86
10^{-1}	1.75	1.65	1.57	1.76	1.87	1.96	2.30

V. CONCLUSIONS

A nonlinear finite element formalism based on the geometrically exact beam theory is presented to model the dynamics of flexible manipulators. The FE beam element on which it is based exhibits important features in the perspective to use finite elements within the control loop. The method allows simulating serial and parallel robots with both rigid and flexible arms, which can be connected with all kind of joints. The procedure for modelling a mechanism, within the simulator in developing, results straightforward. In order to validate this simulator for the dynamic analysis of flexible manipulators, we presented a two−link mechanism subject to a certain motion.

The availability of efficient and physical−based simulators is particularly important in designing manipulators for applications in challenging environments such as space or reactor vessels where the production's costs of the prototypes is still high. The next step in the research of the authors is to optimize the simulations, exploiting parallel and GPU computing in order to reduce the computational time. After, the simulator will be provided of planning and control capabilities, developing a new class of model−based control systems within this framework.

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